Calculations for the Greenhouse Development Rights Calculator

Eric Kemp-Benedict*, Tom Athanasiou, Paul Baer, and Sivan Kartha

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Abstract

The Greenhouse Development Rights (GDRs) framework allocates responsibility for greenhouse gas mitigation and adaptation based on individual responsibility for historical emissions and capacity to act. The GDRs calculator is a web-based tool for exploring how specific quantitative expressions of the GDRs framework affect allocations. This paper documents the calculations that the calculator implements.

Keywords: UNFCCC, climate, inequality, GHG emissions, lognormal

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*Corresponding author: eric.kemp-benedict@sei-international.org

1 Introduction

The Greenhouse Development Rights (GDRs) framework [1] is a general framework for burden-sharing for climate change that takes both responsibility and financial capacity into account. The framework proposes a quantification of the official principles of the United Nations Framework Convention on Climate Change, which call for "the widest possible cooperation by all countries and their participation in an effective and appropriate international response, in accordance with their common but differentiated responsibilities and respective capabilities."

The creators of the GDRs framework maintain an on-line calculator in order to explore the implications of different quantitative expressions of the framework for emissions allocations.[1] This technical paper describes the calculations that are implemented in the calculator. The calculator "engine" itself is available as a program written in C; the code is open-source, and can be downloaded or browsed online.[2]

1.1 Comparing Economic Output: PPP vs MER

The GDRs framework compares the individuals within the world's diverse economies to each other. Any such comparison depends upon what is compared—that is, which categories in the national accounts—and on what basis—that is, the exchange rate. As a practical matter, the choice of exchange rate is limited to market exchange rates (MER) or purchasing power parities (PPP). The MER conversion factors are essentially prices for currencies, and changes in MER reflect changes in demand for currencies in international exchanges. MER values are available on an almost continuous basis. In contrast, PPP conversion factors are calculated laboriously and infrequently, using detailed data on consumption and local prices [7]. Roughly speaking, MER-converted currencies express purchasing power in international markets, while PPP-converted currencies express purchasing power within countries.

All calculations within the GDRs calculator operate on gross domestic product (GDP). In this paper we use the term "income" as a shorthand for GDP per capita. The calculator uses both MER and PPP conversion factors for different purposes, as explained later in this report, and applies conversion factors at national scale. So, although purchasing power may vary significantly between sectors and sub-national regions [3], the calculator assumes that the distribution of income in PPP terms is the same as that in MER terms.

In the presentation in this report we assume that all incomes are PPP-adjusted and, where appropriate, we explicitly convert to MER-adjusted incomes using the national price level π_i, given by the ratio of MER-adjusted GDP to PPP-adjusted GDP [3], $\pi_i = \mathrm{GDP}_{\mathrm{MER},i}/\mathrm{GDP}_{\mathrm{PPP},i}$. In terms of exchange rates, this is expressed as $\pi_i = \mathrm{PPP}_i/\mathrm{MER}_i$.

Our use of PPP-based figures in this report follows from the historical development of the calculator code. The GDRs framework, which allocates burden-sharing within an international regime, uses MER-denominated currencies in most cases. However, the calculator code, which is described in this report, was written as though all currencies were expressed in PPP terms, and converts to MER as needed. Our convention, in this report, of assuming PPP-adjusted currencies, simplifies comparison to the code.

1.2 Lognormal Income Distributions

Because the GDRs framework takes the individual as the unit of analysis, those in low-income countries with high incomes, and those in high-income countries with low incomes, are assessed on the basis of their own income and corresponding emissions rather than the average income of the country in which they live. This approach requires knowledge of within-country income distributions. The availability, since the mid-1990s, of a reasonably complete international database on income distribution [4, 11] makes this calculation possible.

Following Kemp-Benedict [6] and Lopez and Servén [8], income is assumed to be distributed lognormally within countries. The lognormal distribution has two parameters, the mean income $\bar{y}$ and the standard deviation of the log of income, σ. The standard deviation of the log of income is a measure of how equally

[1] The web interface to the calculator can be accessed at http://www.gdrights.org/calculator/.

[2] The code for the calculator engine, written in C, is licensed under the Apache License Version 2.0, a very permissive open-source license. To download or browse the code, go to http://gdrs.sourceforge.net/. There is also a web API interface to the online calculator. The web interface, including the API, is written in PHP. Those who are interested in the PHP code should contact the first author of this report.

or unequally distributed income is within the country, and can be related to the well-known Gini coefficient G using the following formula,

$$\sigma = \sqrt{2} N^{-1} \left(\frac{1+G}{2} \right). \tag{1.1}$$

In this expression, N^{-1} is the inverse normal function. This equation follows from the Gini coefficient for a sum of lognormals, which is derived in Appendix A, in the special case that there is only one country.

For later convenience we transform income y into a new variable z, given by

$$z = \frac{1}{\sigma} \ln(y/\bar{y}) + \frac{\sigma}{2}. \tag{1.2}$$

In terms of z, the lognormal becomes simply a normal distribution with mean zero and standard deviation equal to one,

$$y \sim \text{Lognormal}(\bar{y}, \sigma) \;\Rightarrow\; z \sim N(0,1). \tag{1.3}$$

2 Capacity and Responsibility

The GDRs framework proposes two different income thresholds, a lower threshold y_l, called the "development threshold", and an upper threshold y_u, called the "luxury threshold". Income is exempt below the development threshold, while above the luxury threshold 100 per cent of income contributes to capacity. In the default implementation of the GDRs framework in the calculator, 100 per cent of income above the development threshold also counts toward capacity. Optionally, between the two thresholds a rising fraction $\varphi(y)$ of each marginal increment of income contributes toward capacity, where $\varphi(y)$ increases from 0 at the lower threshold to 100 per cent at the upper threshold. The development threshold reflects purchasing power within countries at low income levels, and within the calculator is compared to PPP-adjusted incomes. In contrast, the luxury threshold applies to high-income consumption that occurs largely, although not entirely, within the global economy, and is compared to MER-adjusted incomes. Thus, the fraction of income that contributes toward capacity between the thresholds is, optionally, equal to

$$\varphi(y) = \frac{\pi_i y - \pi_i y_l}{y_u - \pi_i y_l},\; \pi_i y_l \leqslant \pi_i y \leqslant y_u, \tag{2.1}$$

where we have converted income, y, and the lower threshold y_l from PPP to MER terms.

The contributions to capacity at different income levels are applied like tax rates, so they distinguish between expenditure categories for each individual, rather than between low-income and high-income individuals within a country. We imagine each person (or, more likely, a household) creating a budget, ranking expenditures from least negotiable to most negotiable, and assigning each unit of income first to the least-negotiable expenditures and then increasingly to negotiable ones. At the low end of the scale, we argue that the relevant comparison is purchasing power within the country in which that person or household lives. At the upper end of the scale, the decision to use PPP or MER conversion rates is more difficult, as luxury expenditure might be on non-traded services or on traded goods. Short of calculating a weighted average of PPP and MER ratios, for which we have limited empirical justification, we are left with choosing one conversion factor or the other; for high-end consumption, comparing purchasing power in international markets appears to us the most apt.

2.1 Emissions elasticities and standard integrals

As noted above, the GDRs framework proposes that income below a certain lower income threshold y_l is exempt, in that it does not contribute to an individual's capacity to pay. Also, emissions associated with incomes below y_l are exempt, in that they do not contribute to an individual's historical responsibility. Specifically, the calculator estimates individual emissions at income y as

$$\epsilon_i(y) = \hat{r}_i y^{\gamma}, \tag{2.2}$$

where $\epsilon_i(y)$ represents emissions per capita for individuals in a narrow range of incomes around y for country i, and $\hat{r}_i$ is a country-specific constant. The elasticity γ is the same for all countries; Chakravarty et al.

[2] report evidence that this assumption stands up reasonably well to empirical test, and that the data are consistent with γ between about 0.7 and 1.0. The calculations in this report therefore make repeated use of integrals like the following,

$$M_\gamma(y_c, \sigma, \bar{y}) \equiv \int_{y_c}^{\infty} dy\ y^\gamma f(y; \bar{y}, \sigma), \tag{2.3}$$

where $f(y; \bar{y}, \sigma)$ is the lognormal income distribution, and y_c is an income cutoff. Changing variables using Equation (1.2), the integral becomes

$$M_\gamma(y_c, \sigma, \bar{y}) = \bar{y}^\gamma e^{-\gamma\sigma^2/2} \int_{z_c(y_c)}^{\infty} dz\ e^{\gamma\sigma z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}. \tag{2.4}$$

After combining the exponents, completing the square, and shifting the endpoint of the integral, the result is

$$M_\gamma(y_c, \sigma, \bar{y}) = \bar{y}^\gamma e^{\frac{\sigma^2}{2}\gamma(\gamma-1)} \left[1 - N\left(z_c - \gamma\sigma\right)\right], \tag{2.5}$$

where $N(\cdot)$ is the cumulative normal distribution and z_c is the value of z, from Equation (1.2), corresponding to mean income $\bar{y}$, standard deviation of the log of income σ, and income cutoff y_c. Note that because z_c does not change if both y_c and $\bar{y}$ are rescaled by the same factor,

$$M_\gamma(sy_c, \sigma, s\bar{y}) = s^\gamma M_\gamma(y_c, \sigma, \bar{y}). \tag{2.6}$$

2.2 Individual capacity and responsibility per capita

Capacity to pay for climate mitigation and adaptation is, in the calculator, expressed in MER terms. While this choice imperfectly reflects the baskets of goods that would ultimately correspond to emergency mitigation and adaptation investments, we argue that it is more relevant than PPP. Low-carbon technologies are often, although not always, high technologies that require specialized facilities and skills and are traded internationally, and any global fund for either mitigation or adaptation will of necessity be assessed at market exchange rates.

In the default implementation of GDRs, individual capacity per capita in country i, $c_i(y)$, is given by

$$c_i(y) = \begin{cases} 0 & y < y_l \\ \pi_i(y - y_l) & y > y_l \end{cases}. \tag{2.7}$$

In this expression we have converted incomes, y, which are PPP-denominated in this report, and the PPP-denominated development threshold y_l, to MER terms using the price levels π_i. In the alternative implementation, where capacity changes according to Equation (2.1) between thresholds, we must sum (that is, integrate) capacity for any incomes between the lower and upper thresholds. Above the upper threshold, each dollar contributes fully to capacity and so individual capacity increases linearly above that level. For each small increment dw of income between the lower and upper thresholds, we apply Equation (2.1), so that

$$c_i(y) \ = \ \int_{\pi_i y_l}^{\pi_i y} dw\, \frac{w - \pi_i y_l}{y_u - \pi_i y_l},\ \pi_i y_l \leqslant \pi_i y \leqslant y_u. \tag{2.8}$$

Responsibility is based on emissions above the threshold, where individual emissions per capita vary as in Equation (2.2). In the default implementation of the GDRs framework, the annual contribution to individual per capita responsibility r_i^{ann} is then given by

$$r_i^{\text{ann}}(y) = \begin{cases} 0 & y < y_l \\ \hat{r}_i(y^\gamma - y_l^\gamma) & y \geqslant y_l \end{cases}. \tag{2.9}$$

The coefficient $\hat{r}_i$ in Equation (2.9) can be determined from the total national annual emissions E_i, using the relationship

$$E_i \ = \ \int_0^\infty dy\ \hat{r}_i y^\gamma f(y; \bar{y}_i, \sigma_i) = \hat{r}_i \bar{y}_i^\gamma e^{\frac{\sigma_i^2}{2}\gamma(\gamma-1)}. \tag{2.10}$$

From this equation it can be seen that

$$\hat{r}_i = \frac{E_i}{\bar{y}_i^\gamma} e^{-\frac{\sigma_i^2}{2}\gamma(\gamma-1)}. \tag{2.11}$$

In the alternative implementation, in which responsibility and capacity vary between the development and luxury thresholds, we must sum responsibility between the lower and upper thresholds. Similarly to Equation (2.8), the contribution toward responsibility for y between y_l and y_u, is given by

$$r_i^{\text{ann}}(y) \;=\; \hat{r}_i \int_{y_l^\gamma}^{y^\gamma} dw^\gamma \, \frac{w - y_l}{y_u/\pi_i - y_l}, \quad y_l \leqslant y \leqslant y_u, \tag{2.12}$$

where $\hat{r}_i$ is given by Equation (2.11).

Comparing Equations (2.8) and (2.12), it is seen that the expression for capacity is the same as that for responsibility, with $\gamma = 1$. For this reason, the expression for responsibility is more general. To simplify the later presentation of the explicit expressions for capacity and responsibility, we define some convenient functions, starting with Equation (2.12). First, we define

$$a_\gamma(y; a, b) \equiv \int_{a^\gamma}^{y^\gamma} dw^\gamma \, \frac{w - a}{b - a}. \tag{2.13}$$

This can be evaluated explicitly to give

$$a_\gamma(y; a, b) = \frac{1}{b-a}\left[\frac{\gamma}{\gamma+1}\left(y^{\gamma+1} - a^{\gamma+1}\right) - a\left(y^\gamma - a^\gamma\right)\right]. \tag{2.14}$$

We also define

$$b_\gamma(y; b) = y^\gamma - b^\gamma. \tag{2.15}$$

In terms of the functions $a_\gamma(y; a, b)$ and $b_\gamma(y; b)$, the expression for individual capacity in the alternative implementation becomes

$$c_i(y) = \begin{cases} 0 & \pi_i y < \pi_i y_l, \\ a_1(\pi_i y; \pi_i y_l, y_u) & \pi_i y_l \leqslant \pi_i y \leqslant y_u, \\ a_1(y_u; \pi_i y_l, y_u) + b_1(\pi_i y, y_u) & \pi_i y > y_u, \end{cases} \tag{2.16}$$

while the expression for individual annual responsibility becomes

$$r_i^{\text{ann}}(y) = \hat{r}_i \begin{cases} 0 & y < y_l, \\ a_\gamma(y; y_l, y_u/\pi_i) & y_l \leqslant y \leqslant y_u/\pi_i, \\ a_\gamma(y_u/\pi_i; y_l, y_u/\pi_i) + b_\gamma(y, y_u/\pi_i) & \pi_i y > y_u, \end{cases} \tag{2.17}$$

where $\hat{r}_i$ is given by Equation (2.11).

2.3 National capacity and responsibility

National capacity C_i is given, in the default implementation of the GDRs framework, by integrating per capita capacity, given by Equation (2.7), over the income distribution. Using the formula in Equation (2.5), and rearranging, this can be shown to equal

$$C_i = P_i \left[M_1(\pi_i y_l, \sigma_i, \pi_i \bar{y}_i) - \pi_i y_l M_0(\pi_i y_l, \sigma_i, \pi_i \bar{y}_i)\right], \tag{2.18}$$

where P_i is national population. National annual responsibility R_i^{ann} is found by integrating per capita annual responsibility, given by Equation (2.9), over the income distribution. The expression is closely analogous to the one in Equation (2.18),

$$R_i^{\text{ann}} = \hat{r}_i \left[M_\gamma(y_l, \sigma_i, \bar{y}_i) - y_l^\gamma M_0(y_l, \sigma_i, \bar{y}_i)\right], \tag{2.19}$$

where $\hat{r}_i$ is given by Equation (2.11).

In any given year Y, national responsibility R_i is calculated as cumulative annual responsibility from a starting year t_{ref}. Adding a time index t to annual responsibility, this can be written

$$R_{i,Y} = \sum_{t=t_{\text{ref}}}^{Y} R^{\text{ann}}_{i,t}. \tag{2.20}$$

Unless introduced explicitly, subsequent formulas will suppress the time index.

In the alternative implementation of the framework, the expressions for national capacity and annual responsibility are somewhat more involved than in the default case. Also, in some cases we want to calculate the responsibility or capacity between two income levels $y_a < y_b$. The different cases are shown in Table 1. Also shown in the table are the expressions in terms of the following two functions:

$$A_\gamma(a,b) \equiv A_\gamma(a,b;y_l,y_u/\pi_i,\bar{y}_i,\sigma_i) = \int_a^b dy\, a_\gamma(y;y_l,y_u/\pi_i) f(y;\bar{y}_i,\sigma_i), \tag{2.21a}$$

$$B_\gamma(a,b) \equiv B_\gamma(a,b;y_u/\pi_i,\bar{y}_i,\sigma_i) = \int_a^b dy\, b_\gamma(y;y_u/\pi_i) f(y;\bar{y}_i,\sigma_i). \tag{2.21b}$$

Table 1: Possible relative values of y_a, y_b, y_l, and y_u

	below y_l	between y_l and y_u/π	above y_u/π	expression
1.	$y_a < y_b$			0
2.	y_a	y_b		$A_\gamma(y_l, y_b)$
3.	y_a		y_b	$A_\gamma(y_l, y_u/\pi) + B_\gamma(y_u/\pi, y_b)$
4.		$y_a < y_b$		$A_\gamma(y_a, y_b)$
5.		y_a	y_b	$A_\gamma(y_a, y_u/\pi) + B_\gamma(y_u/\pi, y_b)$
6.			$y_a < y_b$	$B_\gamma(y_a, y_b)$

The expressions for $A_\gamma(a,b)$ and $B_\gamma(a,b)$ given in Equation (2.21) can be written explicitly in terms of the function $M_\gamma(y_c,\sigma,\bar{y})$ introduced in Equation (2.3). Specifically,[3]

$$A_\gamma(a,b) = \frac{1}{y/\pi_i - y_l}\left[\frac{\gamma}{\gamma+1}\left(M_{\gamma+1}(a) - M_{\gamma+1}(b)\right) - y_l\left(M_\gamma(a) - M_\gamma(b)\right) + \frac{y_l^{\gamma+1}}{\gamma+1}\left(M_0(a) - M_0(b)\right)\right], \tag{2.22a}$$

$$B_\gamma(a,b) = \left(M_\gamma(a) - M_\gamma(b)\right) - \left(\frac{y_u}{\pi_i}\right)^\gamma \left(M_0(a) - M_0(b)\right), \tag{2.22b}$$

where we have used the shorthand $M_\gamma(a) \equiv M_\gamma(a,\sigma_i,\bar{y}_i)$.

2.4 The Responsibility-Capacity Indicator

The Responsibility-Capacity Indicator (RCI) is the key indicator for the GDRs framework. It is used to allocate burden-sharing. National RCI, as implemented in the calculator, is a weighted sum of the national share of global responsibility and capacity. That is,

$$\text{RCI}_i = a\frac{R_i}{\sum_{j=1}^N R_j} + (1-a)\frac{C_i}{\sum_{j=1}^N C_j}, \tag{2.23}$$

where N is the number of countries and a lies between zero and one. National RCIs sum to one across countries, and so national RCIs are also the national share of global total RCI. For an individual within country i, RCI per capita, $\text{rci}_i(y)$, is calculated as

$$\text{rci}_i(y) = a\frac{R_i}{\sum_{j=1}^N R_j}\left(\frac{r_i^{\text{ann}}(y)}{R_i^{\text{ann}}}\right) + (1-a)\frac{c_i(y)}{\sum_{j=1}^N C_j}. \tag{2.24}$$

[3]Note that, because the integral that defines $M_\gamma(y_c,\sigma,\bar{y})$ runs from y_c to infinity, it decreases with increasing y_c.

This formulation distributes cumulative responsibility amongst individuals according to the current annual allocations in any given year. This is not ideal, since in fact individuals will have their own income trajectories that diverge from the average trend, depending on the degree of income mobility within the country in which they live. However, the very limited data on income mobility is insufficient to capture this dynamic.

3 Baseline Emissions

Within the GDRs calculator, and consistent with the GDRs framework, baseline emissions are specified for each country as a trajectory over time, $E_i(t)$. The trajectories extend from a past date (for example, 1990 or before), through the start of the "emergency" mitigation program, into the future. Under the GDRs framework, and some other effort-sharing frameworks, countries share the effort of reducing their emissions below their baselines to meet a global target emissions pathway. The GDRs online calculator allocates allowed emissions to each country by subtracting a reduction obligation from its baseline in proportion to its RCI in each year.[4] The baseline is therefore an important input into the GDRs calculator. However, the calculator itself is indifferent to the choice of baseline, so national baselines are documented elsewhere. In this section we discuss two sub-topics that do involve the calculator: consumption-based baselines and luxury-capped baselines.[5]

3.1 Consumption-based baseline emissions

By default, the baselines in the calculator are based on productive activities within national borders. This is consistent with IPCC guidelines for emissions inventories [5] and to the "state of play" within the climate negotiations. However, there is evidence that high-income and high-emitting countries have stabilized their emissions in part by importing carbon-intensive goods from emerging economies [9]. There is no agreement on whether the importer or exporter should bear the responsibility for emissions embodied in traded goods. When the importer bears the responsibility, baseline emissions are consumption-based; when the exporter bears the responsibility, baselines are production-based. To inform the debate, the GDRs calculator allows for consumption-based baseline emissions. As for baselines generally, the calculator is indifferent to the procedure for calculating national net emissions embodied in traded goods, so they are documented elsewhere.

The calculations for consumption-based baseline emissions E_i^{cons} are the same as for production-based baseline emissions E_i^{prod}. The only difference is that the baseline is corrected for net emissions embodied in exports, $\Delta E_i^{\text{exp,net}}$,

$$E_i = E_i^{\text{cons}} = E_i^{\text{prod}} - \Delta E_i^{\text{exp,net}}. \tag{3.1}$$

To facilitate comparison to production-based schemes the calculator presents results in terms of production-based baseline emissions. Because allocations are calculated as a reduction from the baseline, this correction can be accomplished either by: a) subtracting from the production-based baseline, or b) by subtracting from the consumption-based baseline and subsequently adding emissions embodied in exports to the allocation. The calculator follows the second approach.

3.2 Luxury-capped baseline emissions

Using baselines to calculate allocations takes special national circumstances into account, but also implicitly rewards inefficiency and profligate consumption; countries in which a significant part of baseline emissions could be reduced at little cost with relatively minor changes in lifestyle and expectations have an easier time under the GDRs framework than do other countries. To correct for this, the GDRs online calculator offers an option to cap baselines at the upper "luxury" income threshold. When calculating historical emissions, the calculator uses the full baseline or allocated emissions. However, when assigning countries' share of the

[4] Importantly, the GDRs allocation will generally not equal domestic emissions. In some cases allocations are substantially negative, so that it is physically untenable, and economically irrational, for countries to achieve them domestically. Rather, the GDRs allocation says what part of the global emissions reduction a particular country is responsible for, whether it is achieved domestically or abroad.

[5] The calculator also constructs baselines by optionally adding emissions from non-CO_2 gases and emissions from land use, land cover change, and forestry (LULUCF) to the default fossil CO_2 emissions. However, this is a straightforward operation and is not described in this documentation.

burden of reaching the emergency pathway, it uses the lower, luxury-capped baselines, and requires that countries remove their luxury emissions as part of their commitment under the burden-sharing framework.

Luxury-capped baselines are defined in the following way. First, similar to Equation (2.17), for each time step we calculate emissions associated with incomes above the luxury threshold y_u by integrating over individual-level luxury emissions $\lambda_i(y; y_u)$,

$$\lambda_i(y; y_u) = \begin{cases} 0 & y < y_u/\pi_i \\ \frac{B_i}{\bar{y}_i^\gamma} e^{-\frac{\sigma_i^2}{2}\gamma(\gamma-1)} \left(y^\gamma - \frac{y_u^\gamma}{\pi_i^\gamma}\right) & y \geqslant y_u/\pi_i \end{cases}, \tag{3.2}$$

where B_i are baseline emissions. Integrated luxury emissions, $\Lambda_i(y_u)$, across all income levels, are then given by

$$\Lambda_i(y_u) = \frac{B_i}{\bar{y}_i^\gamma} e^{-\frac{\sigma_i^2}{2}\gamma(\gamma-1)} \left[M_\gamma(y_u/\pi_i, \sigma_i, \bar{y}_i) - \left(\frac{y_u}{\pi_i}\right)^\gamma M_0(y_u/\pi_i, \sigma_i, \bar{y}_i)\right]. \tag{3.3}$$

Expressed in terms of the cumulative normal distribution, using Equation (2.5), these are

$$\Lambda_i(y_u) = B_i \left[1 - N(z_{ui} - \gamma\sigma_i) - \left(\frac{y_u}{\pi_i \bar{y}_i}\right)^\gamma e^{-\frac{\sigma_i^2}{2}\gamma(\gamma-1)} \left(1 - N(z_{ui})\right)\right]. \tag{3.4}$$

The luxury-capped baseline emissions $B_i^{\text{lc}}(y_u)$ are then calculated as

$$B_i^{\text{lc}}(y_u) \;=\; B_i - \Lambda_i(y_u) \;=\; B_i \left[N(z_{ui} - \gamma\sigma_i) + \left(\frac{y_u}{\pi_i \bar{y}_i}\right)^\gamma e^{-\frac{\sigma_i^2}{2}\gamma(\gamma-1)} \left(1 - N(z_{ui})\right)\right]. \tag{3.5}$$

It is possible, especially in the early years of the emergency program, that the luxury emissions will exceed the gap between total baseline emissions and the global emergency pathway E_{EP}. One way to think of this situation is that the gradual divergence of the emergency pathway from the baseline represents an exemption of certain luxury emissions in the early years of the program. Accordingly, in this case we adjust the luxury threshold until total luxury-capped emissions equal the emergency pathway. That is, we find an adjusted threshold y_u^* that satisfies

$$\sum_{i=1}^{N} B_i^{\text{lc}}(y_u^*) = E_{\text{EP}}, \tag{3.6}$$

where the sum is over the N countries in the calculator database.

We numerically solve Equation (3.6) in the calculator using the Newton-Raphson root-finding method [10]. We define

$$f(y_u) \equiv \sum_{i=1}^{N} B_i^{\text{lc}}(y_u) - E_{\text{EP}}. \tag{3.7}$$

If $f(y_u)$ is positive with the user-defined luxury threshold, then the emergency pathway lies below the adjusted baselines, and we use the user-defined threshold. Otherwise, the user-defined threshold is too restrictive, and we adjust it until $f(y_u^*) = 0$. For the implementation of the Newton-Raphson method, we need the derivative of $f(y_u)$. The derivative follows from Equation (3.5), and is

$$f'(y_u) = \frac{1}{y_u} \sum_{i=1}^{N} \frac{B_i}{\sigma_i} \left\{n(z_{ui} - \gamma\sigma_i) + \left(\frac{y_u}{\pi_i \bar{y}_i}\right)^\gamma e^{-\frac{\sigma_i^2}{2}\gamma(\gamma-1)} \left[\gamma\sigma_i \left(1 - N(z_{ui})\right) - n(z_{ui})\right]\right\}. \tag{3.8}$$

In this equation, $n(\cdot)$ is the normal probability density, which is also the derivative of the cumulative normal distribution. It is given by

$$n(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}. \tag{3.9}$$

In deriving Equation (3.8) we also used the relation

$$\frac{dz_{ui}}{dy_u} = \frac{1}{\sigma_i y_u}. \tag{3.10}$$

The expression in Equation (3.8) can be expressed consistently by first converting mean income from PPP to MER terms; this is the approach used in the calculator code.

Given an initial value for y_u, y_{u0}, the next value y_{u1} is

$$y_{u1} = y_{u0} - \frac{f(y_{u0})}{f'(y_{u0})}. \tag{3.11}$$

The calculator iterates until the absolute value of $f(y_u)$ is below a threshold, set equal to 10^{-9} MtC/year.

3.3 Kyoto-adjusted baseline emissions

In the default version of the calculator, countries may be rewarded for missing (or rejecting) their Kyoto targets with a higher baseline. The "Kyoto-adjusted baseline" (KAB) corrects for this. A country's KAB—as opposed to its business-as-usual (BAU) baseline—begins as the lower of its 2012 emissions or its Kyoto target, and then rises at the same annual rate of growth (as percent increase per year) as its BAU. The difference between a country's BAU and its KAB is that country's own responsibility. Because the Kyoto-adjusted baseline of country i, which we denote $E_i^{\text{KAB}}(t)$, increases at the same rate as BAU emissions $E_i(t)$, it is calculated simply as

$$E_i^{\text{KAB}}(t) = \frac{E_i^{\text{Kyoto}}}{E_i(t_{\text{EP}})} E_i(t), \tag{3.12}$$

where E_i^{Kyoto} is the country's Kyoto commitment at the start of the Emergency Program, t_{EP}. It is possible, especially near the start of the emergency program, that the global total of Kyoto-adjusted baselines exceeds the emergency pathway; in that case we rescale the gaps between the Kyoto-adjusted baseline and BAU so that the total of the scaled emissions equals the emergency pathway for that year.

The calculator database contains information on the 2012 Kyoto target for each Annex I country as a fraction of the 1990 emissions, f_i^{Kyoto}, and the calculator engine computes Kyoto emissions in 2012 as

$$E_i^{\text{Kyoto}} = f_i^{\text{Kyoto}} E_i(1990). \tag{3.13}$$

The database also stores a flag saying whether the country ratified the Kyoto treaty. Countries that did not ratify the treaty (such as the United States) can be optionally added or removed from the KAB-adjusted allocation calculation.

4 Emissions allocations

Under the GDRs framework, emissions are allocated to countries on the basis of their RCI, and total emissions in any year must be less than the "emergency program" emissions pathway. Although the emergency pathway does not start until a future year, for convenience the calculator defines an historical emergency program pathway that is equal to global historical emissions.

Before presenting different variants of the GDRs emissions allocation algorithm, we first define some symbols and terminology. The emergency pathway is denoted by $E_{\text{EP},t}$, indexed by year t. The ith country's baseline emissions pathway (historical followed by a business-as-usual pathway) is denoted by $B_{i,t}$, while its GDRs emissions allocation is $A_{i,t}$, and its RCI is denoted $I_{i,t}$. In all variants of the allocation algorithm, there are two essential anchoring years: t_{ref}, the historical reference year from which cumulative responsibility is calculated, and t_{EP}, the start of the emergency pathway. Some variants have additional anchoring years.

An interesting complication in the GDRs allocation algorithm is that, from the start of the emergency pathway, national emissions—and therefore cumulative responsibility—become dependent on the GDRs allocations themselves. That is, in the expression for responsibility in Equation (2.17), prior to the start of the emergency program t_{EP}, emissions E_i are equal to baseline emissions $B_{i,t}$, while afterward they are equal to allocations $A_{i,t}$.

4.1 Basic GDRs allocations

In the basic GDRs allocation algorithm, national allocations are calculated as the difference between the national baseline and national share of the global emissions reduction obligation, defined as the gap between business-as-usual emissions and the emergency pathway. Shares of the global obligation are equal to RCI. Therefore,

$$A_{i,t}^{\text{basic}} = B_{i,t} - I_{i,t}^{\text{basic}} \left(\sum_{j=1}^{N} B_{j,t} - E_{\text{EP},t} \right). \tag{4.1}$$

where N is the number of countries. Because RCIs sum to one by virtue of their definition in Equation (2.5), the global allocation is equal to the emergency pathway:

$$\sum_{i=1}^{N} A_{i,t}^{\text{basic}} = \sum_{i=1}^{N} B_{i,t} - \left(\sum_{i=1}^{N} I_{i,t}^{\text{basic}} \right) \left(\sum_{j=1}^{N} B_{j,t} - E_{\text{EP},t} \right) = E_{\text{EP},t}. \tag{4.2}$$

All variants of the GDRs algorithm take Equation (4.1) as their starting point. It contains the two essential ingredients of the GDRs approach to allocating burden-sharing: the quantity to be allocated is the gap between development-as-usual emissions and an emissions path consistent with avoiding dangerous climate change; and national obligations are allocated in proportion to RCI.

4.2 Allocations using luxury-capped or Kyoto-adjusted baselines

As explained above, when using luxury-capped baselines, we explicitly calculate any "luxury" emissions $\Lambda_{i,t}$ associated with individual income above the luxury threshold, where the threshold may have been adjusted to satisfy Equation (3.6) if the total of luxury-capped baseline emissions fall below the emergency pathway. Similarly, for Kyoto-adjusted baselines we calculate the gap between the Kyoto-adjusted baseline and the BAU emissions pathway, possibly scaling the gaps to ensure that the global total of the Kyoto-adjusted baselines does not exceed the emergency pathway.

In either case we find a gap between the national baseline and an adjusted baseline, and when both are applied we calculate the maximum of the two gaps. We subtract that gap from national baselines; otherwise, allocations are calculated as in Equation (4.1),

$$A_{i,t}^{\text{lc}} = B_{i,t} - \Lambda_{i,t} - I_{i,t}^{\text{lc}} \left[\sum_{j=1}^{N} (B_{j,t} - \Lambda_{j,t}) - E_{\text{EP},t} \right]. \tag{4.3}$$

As with the basic allocation, by construction the global allocation is equal to the emergency pathway.

Once the luxury-capped allocation scheme is underway, allocations will begin to diverge from those of basic GDRs, so it is difficult to directly compare the two schemes. However, because the allocations must add up to the same total, and national allocations are not the same, some countries will have higher allocations under the luxury-capped scheme than under the basic scheme, and some will have lower allocations.

To understand the implications of the luxury-capping algorithm, suppose that the first L countries, $L < N$, are the luxury-emitting countries responsible for all of the world's luxury emissions, so that $\Lambda_{i,t} = 0$ for $i > L$. Then, summing over the luxury-emitting countries,

$$\sum_{j=1}^{L} A_{j,t}^{\text{lc}} = \sum_{j=1}^{L} B_{j,t} - \sum_{i=1}^{L} I_{i,t}^{\text{lc}} \left(\sum_{j=1}^{N} B_{j,t} - E_{\text{EP},t} \right) - \left(1 - \sum_{i=1}^{L} I_{i,t}^{\text{lc}} \right) \sum_{j=1}^{L} \Lambda_{j,t}, \tag{4.4}$$

where the sums over luxury emissions $\Lambda_{i,t}$ both extend only from $i = 1$ to L, because for other countries luxury emissions are assumed to be zero. The important term here is the last one. Because total global RCI sums to one, the sum of RCI over the L luxury-emitting countries must be less than (or, perhaps, equal to) one, and so the coefficient on the total luxury emissions is negative (or at least not positive). This means that the total allocation of the luxury-emitting countries is lower than it would be if their baselines were not luxury-capped. Therefore, as a group, luxury-emitting countries must exert more effort when using luxury-capped baselines than in basic GDRs, although individual countries may have lower or higher allocations. The difference between the basic and luxury-capped algorithms becomes smaller as the total RCI of the luxury-emitting countries approaches 100 percent—that is, as that set of countries accounts for more of the world's capacity and responsibility.

4.3 Allocations under sequencing

Basic GDRs, as well as its luxury-capped baseline variant, assumes that all countries enter under the GDRs framework at the same time; while countries differ significantly in their responsibility and capacity, the way that these factors are calculated is the same for all. However, the actual UNFCCC process may extend the country groupings defined in the annexes to the Kyoto Protocol beyond the first commitment period. The GDRs sequencing allocation algorithm allows for a phased introduction of GDRs in which the Annex I countries act first, followed by all countries after a delay.

The sequencing algorithm requires two additional anchoring years: the sequencing base year, t_{sby}, and the end of the sequencing period, t_{send}. The algorithm assumes that the Annex I countries as a whole must reduce their total emissions by a fraction α between t_{sby} and t_{send}, with reductions amongst Annex I countries allocated in proportion to RCI. In keeping with the GDRs framework, global emissions must stay below the emergency pathway; the burden of dealing with the "mitigation gap" between Annex I reductions and the reductions needed to reach the emergency pathway can be assigned either to the Annex II countries alone or to all of the Annex I countries.

4.3.1 Modification to the basic GDRs algorithm

Actual climate mitigation proposals that feature initial action by Annex I countries are underspecified. To create the year-on-year pathway that the GDRs calculator requires, we must specify a shape to the Annex I emissions pathway.

Within the calculator, once the emergency program starts, the sequencing algorithm has two distinct phases: a first sequencing period, when $t_{\text{EP}} < t \leq t_{\text{send}}$, and the second sequencing period, when $t > t_{\text{send}}$. During the first period, the Annex I countries reduce their total emissions in year t relative to their total baseline by an amount

$$S_t = \left(\frac{t - t_{\text{EP}}}{t_{\text{send}} - t_{\text{EP}}}\right)^{\beta} \left[\sum_{i\in \text{A1}} B_{i,t} - (1-\alpha)\sum_{i\in \text{A1}} B_{i,t_{\text{sby}}}\right], \tag{4.5}$$

where the notation "$i \in$ A1" means a sum over all countries within Annex I. While this expression may look obscure, it has the following features:

1. When $t = t_{\text{EP}}$, the Annex I reduction is $S_{t_{\text{EP}}} = 0$.
2. A the end of the first sequencing period, total Annex I emissions are a fraction α below the sequencing base-year emissions,
$$\sum_{i\in \text{A1}} B_{i,t_{\text{EP}}} - S_{t_{\text{EP}}} = (1-\alpha)\sum_{i\in \text{A1}} B_{i,t_{\text{sby}}}.$$
3. The formula interpolates between these two extremes, while the user-defined "smoothing" parameter β ensures that changes in emissions are not too abrupt. By default, β is set equal to two in the calculator.

This list of features reveals Equation (4.5) to be a relatively simple interpolating function that converts an emissions reduction proposal of the form "The Annex I countries will reduce their emissions by a fraction. . . in year. . . below their emissions in the year. . . " into an annual emissions reduction pathway.

The total Annex I reduction is allocated to Annex I countries in proportion to RCI. This gives an initial sequencing allocation of

$$\text{For } t_{\text{EP}} \leqslant t \leqslant t_{\text{send}}\colon\ A^{\text{seq,init}}_{i,t} = \begin{cases} B_{i,t} - S_t \frac{I_{i,t}}{\sum_{j\in \text{A1}} I_{j,t}} & , i \in \text{A1} \\ B_{i,t} & , i \notin \text{A1} \end{cases}. \tag{4.6}$$

That is, during the first sequencing period, non-Annex I countries are permitted to emit at their baseline levels, while Annex I countries reduce their emissions in proportion to their RCI.

The reductions in Equation (4.6) are insufficient to meet the emergency pathway, and so further reductions are required. The mitigation gap at time t, M_t, is given by

$$M_t = \sum_{i=1}^{N} A_{i,t}^{\text{seq,init}} - E_{\text{EP},t}. \tag{4.7}$$

That is, it is the difference between the initial allocations and the emergency pathway. As mentioned above, this mitigation gap might be borne by all Annex I countries, or by the Annex II countries alone. Thus, for Annex I (A1) or Annex II (A2) countries, the final allocation is given by

$$\text{For } t_{\text{EP}} \leqslant t \leqslant t_{\text{send}} \text{ and } i \in \text{An: } A_{i,t}^{\text{seq}} = A_{i,t}^{\text{seq,init}} - \frac{I_{i,t}}{\sum_{j\in\text{An}} I_{j,t}} M_t, \tag{4.8}$$

where An is either A1 or A2. For all other countries the final allocation is equal to the initial allocation.

At the end of the first sequencing period, the calculation shifts towards a variant on the basic GDRs allocation scheme in Equation (4.1). However, Equation (4.1) cannot be applied directly because it would result in an abrupt increase in emissions obligation for non-Annex I countries and a sharp drop for Annex I countries. The calculator ensures a reasonably smooth transition by "freezing" reductions at the end of the first sequencing period, calculating frozen reduction $F_{i,t}$ as

$$F_i \equiv B_{i,t_{\text{send}}} - A_{i,t_{\text{send}}}^{\text{seq}}. \tag{4.9}$$

Note that for non-Annex I countries the frozen reductions are zero. During the second sequencing period (which extends indefinitely), allocations are calculated as

$$\text{For } t > t_{\text{send}}\text{: } A_{i,t}^{\text{seq}} = (B_{i,t} - F_i) - I_{i,t}^{\text{basic}} \left[\sum_{j=1}^{N} (B_{j,t} - F_j) - E_{\text{EP},t} \right]. \tag{4.10}$$

This is the same as the basic GDRs allocation calculation except that Annex I baselines subtract the reductions at the end of the first sequencing period.

4.3.2 Modification to the luxury-capped baseline algorithm

The luxury-capped baseline algorithm is modified analogous to the basic GDRs algorithm. As with basic GDRs, non-Annex I countries are exempt from entering under the GDRs framework until after the first sequencing period. Accordingly, luxury emissions are calculated as in Equation (3.4) for Annex I countries, but are set equal to zero for non-Annex I countries until the second sequencing period. That is,

$$\text{For } t_{\text{EP}} \leqslant t \leqslant t_{\text{send}}\text{: } \Lambda_{i,t}^{\text{seq}}(y_u) = \begin{cases} \Lambda_{i,t}(y_u) & , i \in \text{A1} \\ 0 & , i \notin \text{A1} \end{cases}, \tag{4.11}$$

where $\Lambda_{i,t}(y_u)$ is the time-varying equivalent of Equation (3.4). In the second sequencing period, luxury emissions are calculated as

$$\text{For } t > t_{\text{send}}\text{: } \Lambda_i(y_u) = (B_{i,t} - F_i) \left[1 - N(z_{ui,t} - \gamma\sigma_{i,t}) - \left(\frac{y_u}{\pi_{i,t}\bar{y}_{i,t}} \right)^{\gamma} e^{-\frac{\sigma_{i,t}^2}{2}\gamma(\gamma-1)} \left(1 - N(z_{ui,t})\right) \right]. \tag{4.12}$$

This is the same as in Equation (3.4) except baselines are adjusted by the "frozen" emissions carried over from the end of the first sequencing period, as defined in Equation (4.9).

In all other respects, the calculations are as for sequencing with unadjusted baseline emissions, with one difference. In the first sequencing period, baselines are adjusted by subtracting luxury emissions, as in Equation (4.3), while in the second sequencing period, baselines are adjusted by subtracting both luxury emissions and frozen emissions.

5 Conclusion

This technical paper presents the core calculations used by the Greenhouse Development Rights (GDRs) online calculator. The calculations document the algorithms implemented in the GDRs calculator engine, available as an open-source C program.

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A Gini Coefficient for a Sum of Lognormals

When preparing data for the GDRs calculator, values for China are calculated using data on income distributions and populations for Hong Kong and Mainland China. The combined Gini coefficient for China is estimated as the Gini coefficient for a weighted sum of lognormal distributions, one for Hong Kong and one for Mainland China. The general result for the Gini coefficient of a weighted sum of lognormals is presented in this annex.

A.1 Statement of the result

Suppose that a region is composed of N countries, labeled by $i = 1, \ldots, N$. Each of the N countries is assumed to have lognormally distributed income, with income distributions $f_{\ln}(y, \bar{y}_i, \sigma_i)$, where $f_{\ln}(\cdot)$ is the lognormal distribution, y is per capita income, $\bar{y}_i$ is mean per capita income for country i, and σ_i is the square root of the variance of the logarithm of income. Furthermore, suppose that the share of regional population within each country is s_i, where $\sum_{i=1}^{N} s_i = 1$. Then the regional income distribution $f_{\text{reg}}(y)$ is

$$f_{\text{reg}}(y) = \sum_{i=1}^{N} s_i f_{\ln}(y, \bar{y}_i, \sigma_i), \tag{A.1}$$

regional mean income is

$$\bar{y} = \sum_{i=1}^{N} s_i \bar{y}_i. \tag{A.2}$$

and the regional Gini coefficient, G, is

$$G = 1 - \frac{2}{\bar{y}} \sum_{i=1}^{N} \sum_{j=1}^{N} s_i s_j I_{ij}, \tag{A.3}$$

where

$$I_{ij} = \bar{y}_j N \left[\frac{1}{\sqrt{\sigma_i^2 + \sigma_j^2}} \left(\ln \frac{\bar{y}_i}{\bar{y}_j} + \frac{1}{2}\sigma_i^2 - \frac{3}{2}\sigma_j^2 \right) \right], \tag{A.4}$$

and $N(\cdot)$ is the cumulative standard normal distribution.

A.2 Derivation of the result

This section presents a derivation of Equations (A.3, A.4). The derivation starts with a general expression for the Gini coefficient as a functional (that is, a function of a function) of the income distribution.

A.2.1 Gini coefficient in terms of the income distribution

The Gini coefficient is defined in terms of the Lorenz curve, which is a plot of cumulative income against cumulative population, where the population is ranked in order of increasing income. If the Lorenz curve is represented by $L(x)$, where x is the share of cumulative population, then the Gini coefficient is

$$G = 1 - 2 \int_0^1 L(x)\,dx. \tag{A.5}$$

The integral in Equation (A.5) can be evaluated by carrying out a change of variables, from cumulative population x to income y. They are related through the income distribution. For a generic income distribution $f(y)$,

$$x = \int_0^y dy'\ f(y'). \tag{A.6}$$

Cumulative income—that is, the Lorenz curve—can then be calculated as

$$L(x) = \frac{1}{\bar{y}} \int_0^{y(x)} dy' \; y' f(y'), \tag{A.7}$$

where the dependence of the upper limit $y(x)$ of the integral through Equation (A.6) is shown explicitly. By changing to y as the variable in the integrand, Equation (A.5) can be seen to be equivalent to

$$G = 1 - \frac{2}{\bar{y}} \int_0^{\infty} dy \; f(y) \int_0^{y} dy' \; y' f(y'). \tag{A.8}$$

This is a general result, for any income distribution $f(y)$. In the next section it is applied to the regional income distribution given in Equation (A.1).

A.2.2 Gini coefficient for a weighted sum of lognormal distributions

Applying Equation (A.8) to the regional income distribution in Equation (A.1) gives

$$G = 1 - \frac{2}{\bar{y}} \sum_{i=1}^{N} \sum_{j=1}^{N} s_i s_j I_{ij}, \tag{A.9}$$

that is, Equation (A.3), where

$$I_{ij} = \int_0^{\infty} dy \; f_{\ln}(y, \bar{y}_i, \sigma_i) \int_0^{y} dy' \; y' f_{\ln}(y', \bar{y}_j, \sigma_j). \tag{A.10}$$

The double integral in Equation (A.10) is difficult to evaluate as it is written. It can be transformed using the following procedure. First, define

$$\hat{I}_{ij}(a) = \int_0^{\infty} dy \; f_{\ln}(y, \bar{y}_i, \sigma_i) \int_0^{ay} dy' \; y' f_{\ln}(y', \bar{y}_j, \sigma_j). \tag{A.11}$$

This function is almost identical to I_{ij}, except that a new variable a multiplies the upper limit on the inner integral over y'. This function has the properties that

$$\hat{I}_{ij}(0) = 0 \text{ and } \hat{I}_{ij}(1) = I_{ij}. \tag{A.12}$$

Taking the derivative of $\hat{I}_{ij}(a)$ with respect to a reduces Equation (A.11) to a single integral, but with the consequence that it must later be integrated over a to recover the full function. This turns out to be simpler than evaluating Equation (A.10) directly. The derivative of $\hat{I}_{ij}(a)$ with respect to a is

$$\frac{d\hat{I}_{ij}(a)}{da} = \int_0^{\infty} dy \; ay^2 f_{\ln}(y, \bar{y}_i, \sigma_i) f_{\ln}(ay, \bar{y}_j, \sigma_j). \tag{A.13}$$

This integral is then evaluated by using the standard form for the lognormal income distribution,

$$f_{\ln}(y, \bar{y}, \sigma) = \frac{1}{\sqrt{2\pi}\sigma y} e^{-\frac{1}{2\sigma^2}\left(\ln \frac{y}{\bar{y}} + \frac{\sigma^2}{2}\right)^2}. \tag{A.14}$$

Substituting Equation (A.14) in Equation (A.13), a change of variables from y to $\ln y$ in the integral transforms it into an integral over a product of normal distributions.

Integrals of products of normal distributions can be carried out using standard techniques, by first completing the square in the exponent and then changing variables. The calculation is lengthy, but straightforward, and produces the following result:

$$\frac{d\hat{I}_{ij}(a)}{da} = \frac{1}{\sqrt{2\pi\left(\sigma_i^2 + \sigma_j^2\right)}} \exp\left\{-\frac{1}{2}\frac{1}{\sigma_i^2 + \sigma_j^2}\left[\left(\ln a + A_i - A_j\right)^2 - \sigma_i^2\sigma_j^2 - 2A_i\sigma_j^2 + 2\left(\ln a - A_j\right)\sigma_i^2\right]\right\}, \tag{A.15}$$

where

$$A_k = \ln \bar{y}_k - \frac{\sigma_k^2}{2}. \tag{A.16}$$

The expression for $\hat{I}_{ij}(a)$ itself is found by integrating the right-hand side in Equation (A.15) from zero to one and applying the boundary condition $\hat{I}_{ij}(0) = 0$ from Equation (A.12). This can be done by performing a change of variables from a to $\ln a$, which changes the bounds of the integral from 0 and 1 to $-\infty$ and 0. This results in a standard integral of the normal distribution over half of its domain, and can be done exactly. The result is Equation (A.4).